A Robust Multigrid Solver for Highly Anisotropic and Singular Elliptic Problems on Spheres

Cao Lu, Oliver Yang, Xiangmin Jiao, and Marat Khairoutdinov
Department of Applied Mathematics and Statistics, School of Marine and Atmospheric Sciences, Stony Brook University

The Problem

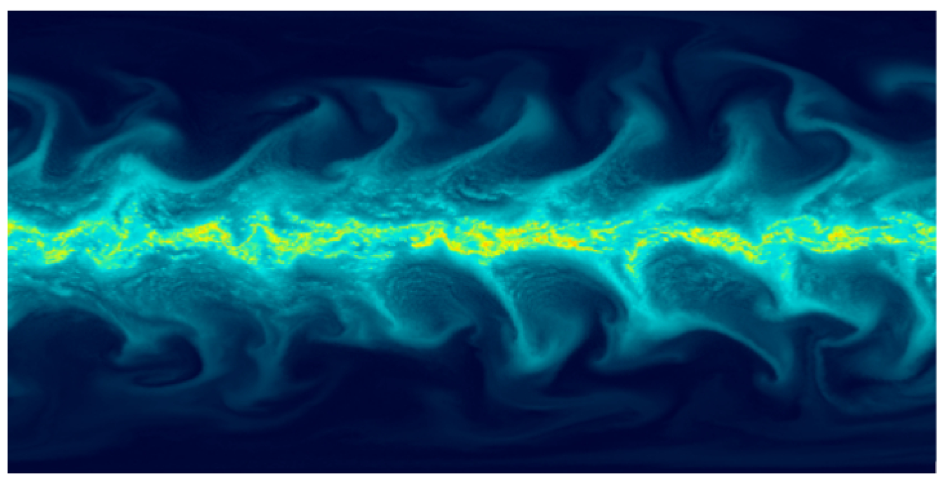

Figure 1: We wish to solve a 3-D Poisson-type equation for the atmospheric pressure. Our solver is used in simulations to understand the effect of clouds on the earth's climate.

Elliptic PDEs with pure Neumann boundary conditions can lead to singular linear systems. Non-uniform grids and varying coefficients also cause non-symmetry. Furthermore, discretization error creates incompatible right-hand-sides, which yield rank-deficient least-squares problems. The adaptation of multigrid to accurately and efficiently solve such problems is a major challenge.

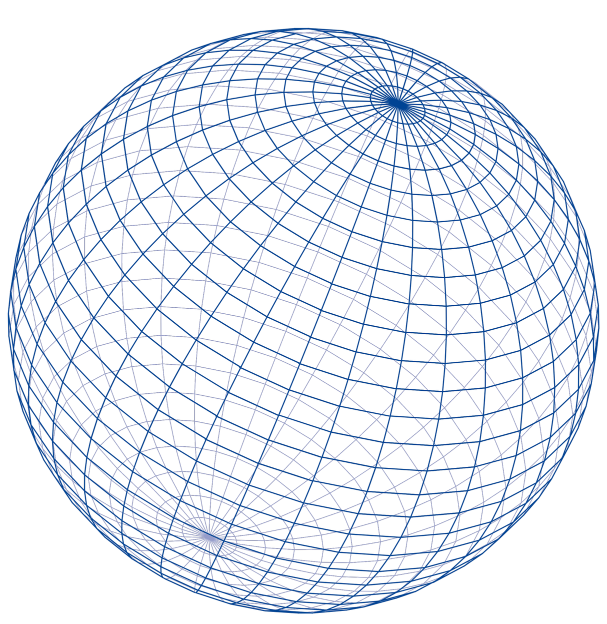

Figure 2: We decompose the 3-D equation via FFT into a collection of 2-D Helmholtz equations.

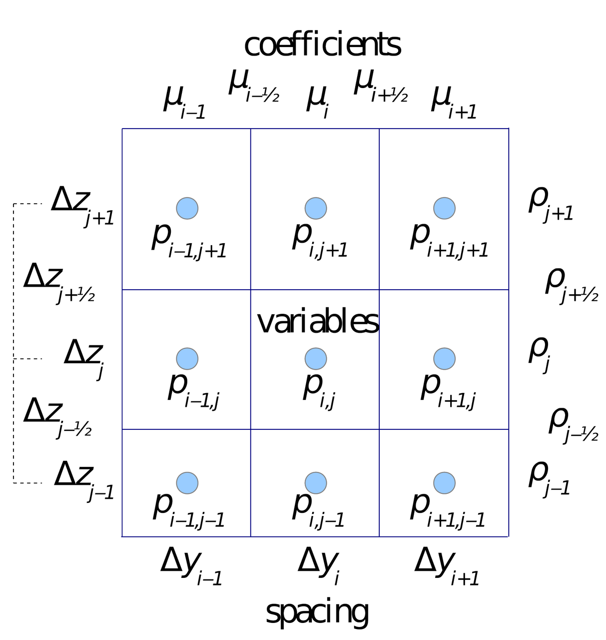

Figure 3: We use a cell-centered, conservative, finite-difference discretization scheme.

Our GMG Solver

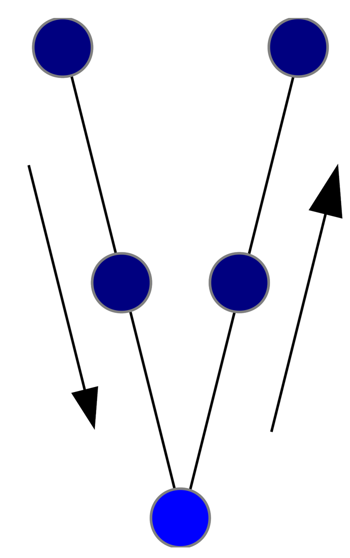

Figure 4: We use V-cycles.

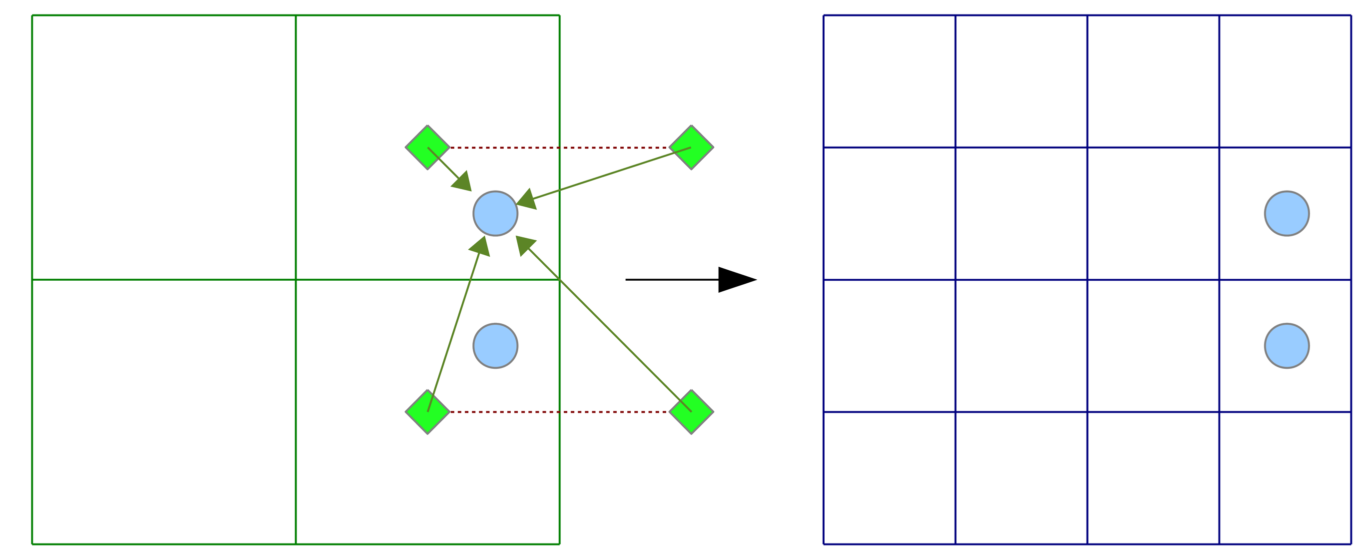

Figure 5: We use bilinear interpolation to derive cell-centered prolongation operators at the boundary (linear along the edges and constant at the corners).

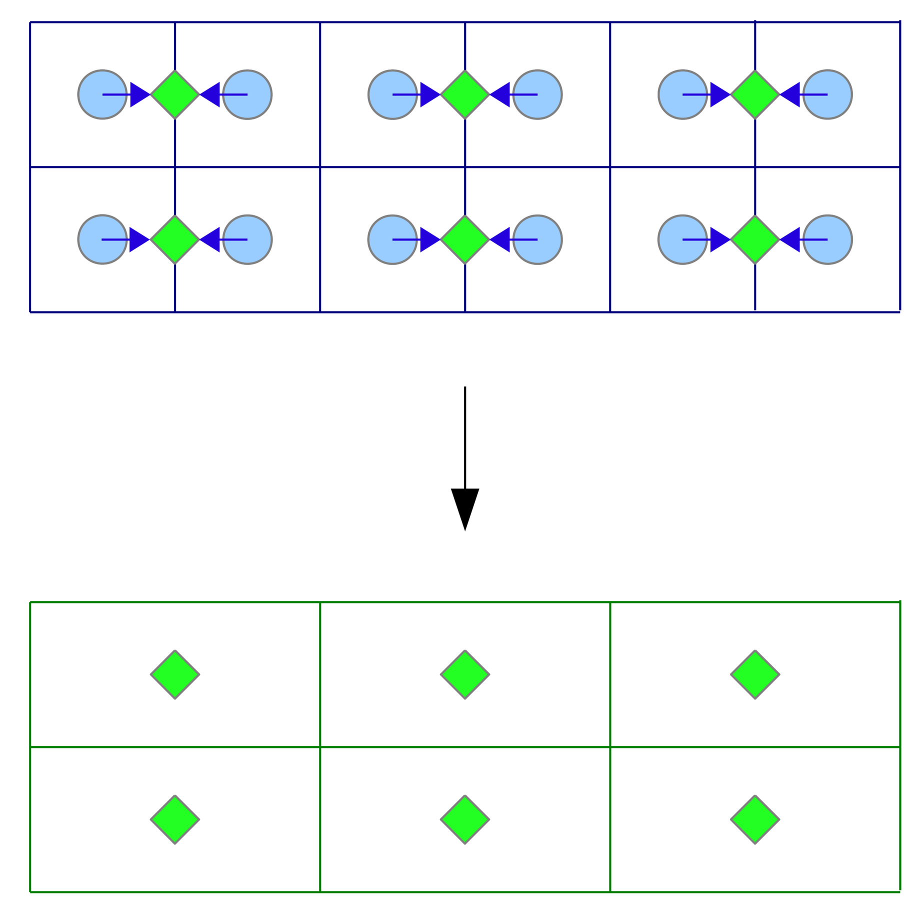

Figure 6: We semi-coarsen to reduce the size of the linear system on the coarsest grid.

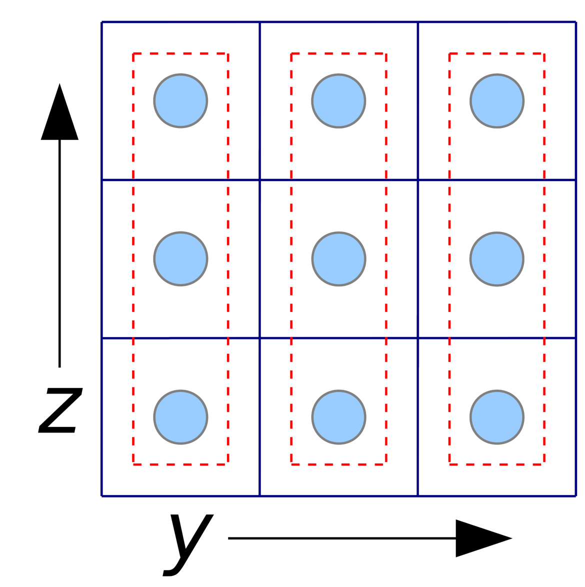

Figure 7: Point-wise smoothers are ineffective since the large degree of anisotropy causes the errors to be geometrically non-smooth. We use line smoothers to update strongly connected unknowns together.

Numerical Results

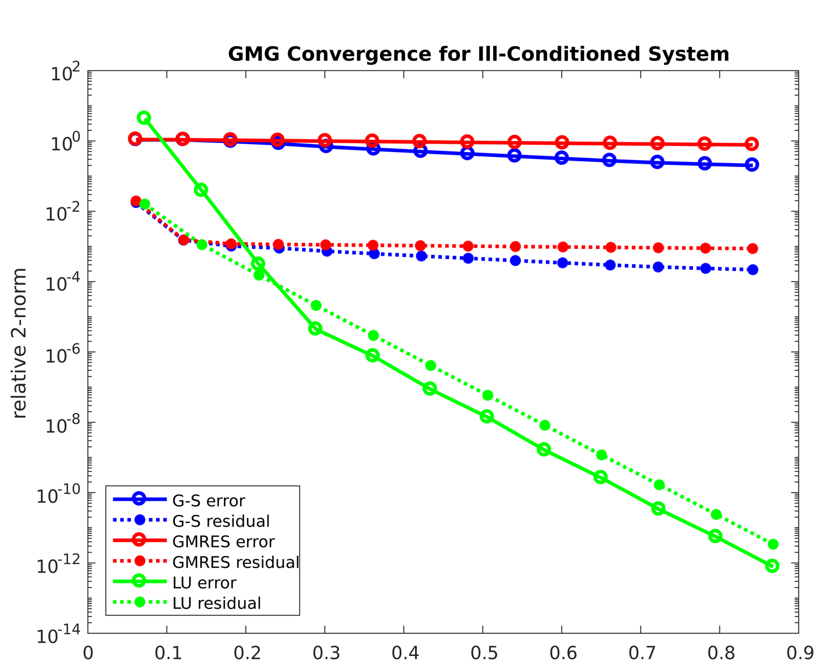

Figure 8: Direct coarse-grid solvers are more accurate and stable for highly ill-conditioned systems.

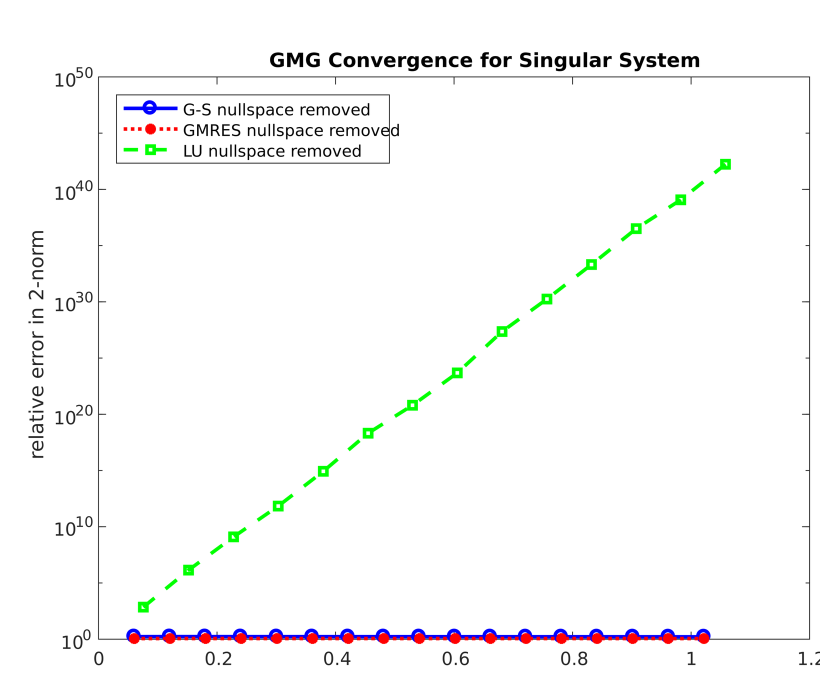

Figure 9: For singular systems, multigrid stagnates or diverges with GMRES or LU as solvers.

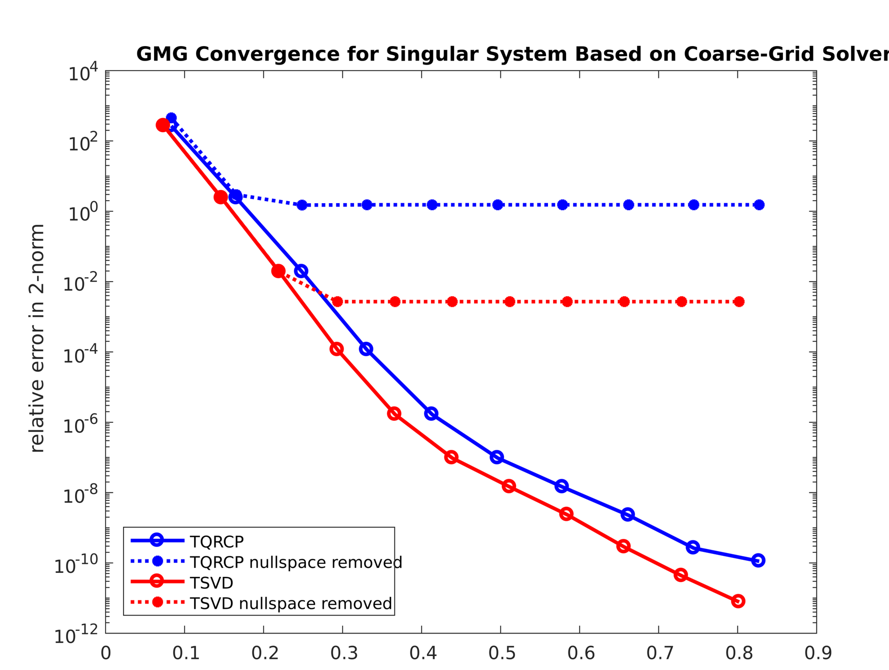

Figure 10: TSVD or TQRCP give efficient and stable convergence to fine-grid least-squares solutions.

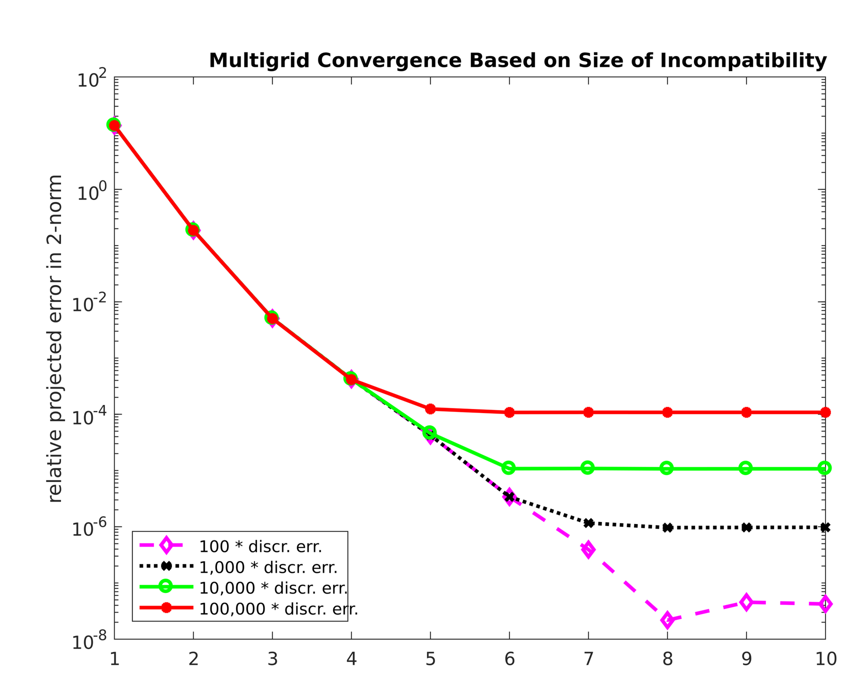

Figure 11: With incompatibility, the solution is perturbed, but maintains the proper order of accuracy.

Rationale for Our Approach

Unlike AMG, GMG rediscretization retains the physical meaning of the original equation on coarser grids. The right nullspace should be a low frequency mode, so it will be resolved on the coarsest grids. All right nullspaces should therefore be similar. Hence, it is reasonable to find and remove the right nullspace of the coarsest grid. Prolongation errors will reintroduce some nullspace component on finer grids, which smoothers will be unable to damp. But this at least allows multigrid to converge to a least-squares solution on the fine-grid with a relatively small nullspace component.

Conclusions

We developed a GMG solver that is robust and much more efficient than state-of-the-art AMG preconditioners. In the singular case, we produced least-squares solutions using TSVD or TQRCP as coarse-grid solvers. We can produce minimum-norm solutions by projecting off the nullspace on the fine-grid, if known.

Acknowledgements

We would like to thank Aditi Ghai for providing us with tuned parameters for Hypre. This work was supported by the NSF.