Theoretical and computational analysis of recurrent neural networks and their bifurcations

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Motivation (I)

(1) The brain acts to control our movements, thoughts, emotions, and regulate bodily processes.

(2) Despite its importance, extremely little is known about how neural computation works.

(3) This is a result of sheer complexity and experimental limitations.

(4) A lack of semantic understanding is a hinderance to furthering the field of neuroscience.
Motivation (II) Why RNNs?

(1) More recently, populations of neurons are assumed to be driven by low-dimensional dynamical systems, bringing rise to computations.

(2) Artificial RNNs are analogous to the recurrent connectivity in the brain.

(3) With artificial RNNs we can work purely constructively.

(4) With artificial RNNs we have access to every parameter in the model.

(5) Experiments with software avoid the safety issues associated with experimentation on wetware.
A brief introduction to dynamical systems

Concepts:
(1) Flow (vector field)
(2) Fixed Point
(3) Topological Stability Structure
(4) Bifurcations and parameter space
(I) Expressive power of continuous-time recurrent neural networks
(II) How RNNs use their underlying topological structure in the context of computations
(III) Future Work: Crossover to spiking neural networks
Expressive power of recurrent neural networks

The main idea: RNNs are intrinsically a discrete approximation of an underlying continuous-time dynamical system.

Real-world problems are, for the most part, continuous. RNNs are “attempting” to approximate these underlying continuous dynamics.

General Form: $h_{t+1} = f(h_t, x_t)$ where $x_t$ is the current input in a sequence indexed by $t$, $f$ is a nonlinear function.

Autonomous Form: $h_{t+1} = f(h_t)$ where $f(\cdot) := f(\cdot, 0)$.
Gated Recurrent Units (GRUs)

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\begin{align*}
    z_t &= \sigma(W_x x_t + U_z h_{t-1} + b_z) & \text{(update gate)} \\
    r_t &= \sigma(W_x x_t + U_r h_{t-1} + b_r) & \text{(reset gate)} \\
    h_t &= (1 - z_t) \odot \tanh(W_h x_t + U_h (r_t \odot h_{t-1} + b_h)) + z_t \odot h_{t-1} & \text{(hidden state)}
\end{align*}
\]

\[x_t \in \mathbb{R}^p, \ h_t \in \mathbb{R}^d\]

where \(W_z, W_r, W_h \in \mathbb{R}^{d \times p}\) and \(U_z, U_r, U_h \in \mathbb{R}^{d \times d}\) are the parameter matrices, \(b_z, b_r, b_h \in \mathbb{R}^d\) are bias vectors, \(\odot\) represents the Hadamard product, and \(\sigma(z) = 1/(1 + e^{-z})\) is the element-wise logistic sigmoid function. Note that the hidden state is asymptotically contained within \([-1, 1]^d\) due

\[
\begin{align*}
    z(t) &= \sigma(U_z h(t) + b_z) & \text{(continuous update gate)} \\
    r(t) &= \sigma(U_r h(t) + b_r) & \text{(continuous reset gate)} \\
    \dot{h} \odot (1 - z(t)) &= -h(t) + \tanh(U_h (r(t) \odot h(t)) + b_h) & \text{(continuous hidden state)}
\end{align*}
\]

where \(\odot\) denotes point-wise division and \(\dot{h} \equiv \frac{dh(t)}{dt}\).
Gated recurrent units viewed through the lens of continuous time dynamical systems

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\begin{align*}
\dot{x} &= -\frac{1}{2}(x - \tanh\left[\frac{3}{2}(\cos(\alpha)x - \sin(\alpha)y - \sin(\alpha)z)\right]) \\
\dot{y} &= -\frac{1}{2}(y - \tanh\left[\frac{3}{2}(\sin(\alpha)x + \cos(\alpha)y + \sin(\alpha)z)\right]) \\
\dot{z} &= -\frac{1}{2}(z - \tanh\left[\frac{3}{2}(\sin(\alpha)x - \sin(\alpha)y + \cos(\alpha)z)\right])
\end{align*}
\]

(i) $\alpha = \frac{\pi}{4}$ 
(ii) $\alpha = \frac{\pi}{4.95956}$ 
(iii) $\alpha = \frac{\pi}{5}$
Part (II)
How do RNNs use their underlying topological structure in the context of computations?
2-sequence classification problem (non-uniqueness and guiding principles)
What about n-sequence?
Part (III)
Future Work: Crossover to spiking neural networks and global brain dynamics
Thank you!

CATNIP Lab